

Mahler measures, K-theory and values of L-functions

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Abstract The Mahler measure of a polynomial P in n variables is defined as the mean of $\log |P|$ over the n -dimensional torus. For certain polynomials with integer coefficients in two variables the Mahler measure is known to be related to special values of L-functions of arithmetic objects (e.g. Dirichlet characters and elliptic curves over \mathbb{Q}). Inspired by work of Deninger ([11]) Boyd has investigated this relationship numerically ([7]). In this paper we reduce some conjectures of Boyd to Beilinson's conjectures on special values of L-functions. The methods in use are widely of K-theoretical nature.

0 Introduction

The (logarithmic) Mahler measure of a polynomial $P \in \mathbb{C}[t_1, t_2]$ is defined as

$$\begin{aligned} m(P) &:= \frac{1}{(2\pi i)^2} \int_{T^2} \log |P(z_1, z_2)| \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \\ &= \int_0^1 \int_0^1 \log |P(e^{2\pi i \alpha_1}, e^{2\pi i \alpha_2})| d\alpha_1 d\alpha_2 \end{aligned}$$

where $T^2 := S^1 \times S^1 \subset \mathbb{C}^2$ is the real 2-torus.

In [17] Smyth discovered the identity

$$(1) \quad m(t_1 + t_2 + 1) = L'(\chi, -1)$$

where χ is the quadratic character of conductor 3 and $L(\chi, s)$ is the Dirichlet series associated to it. Some similar formulas can be found in [6] and [14]. The proofs of these identities however are analytical and do not shed much light on the deeper reasons for this phenomenon.

This was the situation until Deninger in [11] related formulas like (1) to Beilinson's conjectures on special values of L-functions. Assuming these conjectures he found in some way higher dimensional analogues of (1) such as

$$(2) \quad m(t_1^2 t_2 + t_1 t_2^2 + t_1 t_2 + t_1 + t_2) = *L'(E, 0)$$

where $*$ denotes (throughout the whole paper) an unknown non-vanishing rational number and $L(E, s)$ is the Hasse-Weil L-function of the elliptic curve E/\mathbb{Q} obtained by taking the projective closure of the zero locus

$$t_1^2 t_2 + t_1 t_2^2 + t_1 t_2 + t_1 + t_2 = 0$$

and adding a suitable origin.

This example was the starting point for extensive numerical computations done by Boyd (see [7]). He found (numerically) hundreds of formulas like (2) and similar ones. He also stated a condition under the presence of which formulas of type (2) should hold. Rodriguez-Villegas showed in [15] that it is precisely this condition that makes it possible to apply Beilinson's conjectures. For a special class of polynomials this was (up to integrality questions) independently done by the author (see chapter 2).

In this paper we set forth the ideas of [11] and try to interpret further parts of the work of Boyd in the light of Beilinson's conjectures. We succeed in the following cases:

- Boyd observes that some (irreducible) polynomials produce formulas of mixed type, i.e. the Mahler measure of such a polynomial is equal to

$$*L'(\chi, -1) + *L'(E, 0)$$

for some Dirichlet character χ and some elliptic curve E over \mathbb{Q} . For this topic see chapter 4 and 5.

- Another conjecture of Boyd states that no formula of mixed type will occur as long as the polynomial is reciprocal. For this problem see chapter 1.
- Boyd also found formulas of type (2) where the zero locus of the polynomial is of genus two. In those cases the elliptic curve E in (2) turns out to be one of the (generally) non-isogenous factors of the Jacobian of the zero locus. See chapter 3 for an explanation for this rather "miraculous" occurrence.

These notes represent a shortened version of the author's thesis [5]. The reader who wants to see detailed proofs rather than (just) the underlying basic ideas is referred to this work.

Further work in the spirit of [11] was done in the following papers: In [18] the three variable example

$$m(1 + t_1 + t_2 + t_3) = *\zeta'(-2)$$

of Smyth was reduced to the (due to Borel) known Beilinson conjectures for $\text{Spec } \mathbb{Q}$. In [4] an approach to p-adic analogs of Mahler measures and formulas of type (2) can be found.

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Contents

1. Mahler measures and symbols
2. Boundary maps in K -theory
3. Curves of genus 2
4. Formulas of mixed type
5. A general philosophy

1 Mahler measures and symbols

In this chapter we will rewrite the Mahler measure of a polynomial in two variables in a way that allows us to apply K -theoretical methods. The main idea of the following is that in building the Mahler measure of a polynomial by definition we have to integrate over a differential form which can be seen as a certain cup product lying in some Deligne cohomology group. In our context the main lemma is

Lemma 1.1 *For $n \geq 0$ consider elements*

$$\varepsilon_0, \dots, \varepsilon_n \in H_D^1(X, \mathbb{R}(1)) = \left\{ \varepsilon \in \mathcal{A}^0(X, \mathbb{R}) \left| \begin{array}{l} d\varepsilon = \pi_0(\omega), \\ \omega \in \Omega_D^1(\overline{X}) \end{array} \right. \right\}$$

Define a smooth $\mathbb{R}(n)$ -valued n -form on X by:

$$\begin{aligned} C_{n+1} &= C_{n+1}(\varepsilon_0, \dots, \varepsilon_n) \\ &= 2^n \sum_{i=0}^n (-1)^i \frac{1}{(n+1)!} \sum_{\sigma \in \mathfrak{S}_{n+1}} \text{sgn}(\sigma) \varepsilon_{\sigma_0} \bar{\partial} \varepsilon_{\sigma_1} \dots \bar{\partial} \varepsilon_{\sigma_i} \partial \varepsilon_{\sigma_{i+1}} \dots \partial \varepsilon_{\sigma_n}. \end{aligned}$$

where \mathfrak{S}_{n+1} is the permutation group of $\{0, \dots, n\}$. Then

$$dC_{n+1} = \pi_n(\omega_{n+1})$$

where $\omega_{n+1} = 2^{n+1} \partial \varepsilon_0 \wedge \dots \wedge \partial \varepsilon_n \in \Omega_D^{n+1}(\overline{X})$ and

$$[C_{n+1}(\varepsilon_0, \dots, \varepsilon_n)] = [\varepsilon_0] \cup \dots \cup [\varepsilon_n] \text{ in } H_{\mathcal{D}}^{n+1}(X, \mathbb{R}(n+1)).$$

Moreover for all $\sigma \in \mathfrak{S}_{n+1}$

$$C_{n+1}(\varepsilon_{\sigma_0}, \dots, \varepsilon_{\sigma_n}) = \text{sgn}(\sigma) C_{n+1}(\varepsilon_0, \dots, \varepsilon_n).$$

Proof: See [3] 2.2 and [10] Lemma (7.2). \square

Before we can proceed we have to fix some notations. Let $0 \neq P(t_1, t_2) \in \mathbb{C}[t_1, t_2]$,

$$(3) \quad P(t_1, t_2) = \sum_{i=0}^n a_i(t_1) t_2^i$$

be irreducible with $a_n \neq 0$. Set $i_0 := \min\{i \mid a_i \neq 0\}$ and let $P^*(t_1)$ denote the polynomial $a_{i_0}(t_1)$. Assume that $P^*(t_1) = P(t_1, 0)$ and that P^* does not vanish on S^1 . Let $Z^*(P) := Z(P) \cap (\mathbb{C}^*)^2$. Denote by A the union of the connected components of dimension 1 of $(S^1 \times B) \cap Z^*(P) = (S^1 \times B) \cap Z(P)$. Furthermore let $A \subset Z^*(P)^{\text{reg}}$.

As was remarked by Deninger (see [11]) and others using Jensen's formula one has

$$m(P^*) - m(P) = \int_{S^1} \eta$$

with the integrable 1-form on S^1

$$\eta := \frac{1}{2\pi i} \sum_{\substack{0 \neq b \in \mathring{B} \\ P(t_1, b) = 0}} \log |b| \frac{dt_1}{t_1}.$$

The sum has to be taken with multiplicities of the zeroes $0 \neq b \in \mathring{B}$ of $P_{t_1}(t) := P(t_1, t)$. The form η is well defined since $P_{t_1}(t)$ cannot vanish identically due to the irreducibility of P .

Proceeding in the line of [9] Thm. 5.1 we now “triangulate” the compact, semi-algebraic set A . Set

$$\begin{aligned} e : [0, 1] &\rightarrow S^1 \\ \varphi &\mapsto e^{2\pi i \varphi}. \end{aligned}$$

Using implicit functions one can subdivide the interval $I := [0, 1]$ into disjoint subintervals $I_k := [\tau_k, \tau_{k+1}]$ for $k = 0, \dots, s-1$ and define algebraic germs F_{1k}, \dots, F_{nk} of P in a neighbourhood of the arc $e(\overset{\circ}{I}_k)$ which can be continuously extended to the boundary $\partial e(I_k)$. Therefore we have paths

$$\begin{aligned}\gamma_{ik} : I_k &\rightarrow S^1 \times \mathbb{P}^1(\mathbb{C}) \\ \varphi &\mapsto (e(\varphi), F_{ik}(e(\varphi)))\end{aligned}$$

which by eventually taking a finer subdivision have the following properties

1. For a path γ_{ik} one and only one of the following conditions holds:

- (a) $\gamma_{ik}(\overset{\circ}{I}_k) \subset S^1 \times \overset{\circ}{B}$.
- (b) $\gamma_{ik}(\overset{\circ}{I}_k) \subset T^2$.
- (c) $\gamma_{ik}(\overset{\circ}{I}_k) \subset (S^1 \times B)^c$.

2. One has

$$A = \bigcup_{\substack{i,k \\ \gamma_{ik}(I_k) \subset S^1 \times B}} \gamma_{ik}(I_k).$$

3. If two paths γ_{ik} and $\gamma_{i'k'}$ intersect their intersection is contained in $\gamma_{ik}(\partial I_k) \cap \gamma_{i'k'}(\partial I_{k'})$.

Using this construction we have

Lemma 1.2 *Let P satisfy the general assumptions made at the beginning of this chapter. Let $C_2 = C_2(\log |t_1|, \log |t_2|)$ denote the differential form of 1.1. Then the restriction of C_2 to $Z^*(P)^{reg}$ is defined and we have*

$$(-2\pi i) \int_{S^1} \eta = \sum_{k=0}^{s-1} \sum_{\substack{i \in \{1, \dots, n\} \\ \gamma_{ik}(I_k) \subset S^1 \times B}} \int_{I_k} \gamma_{ik}^* C_2.$$

Proof: Since $Z(P) \cap T^2$ doesn't contribute to the integral we have

$$(-2\pi i) \int_{S^1} \eta = (-2\pi i) \int_0^1 \sum_{\substack{0 \neq b \in B \\ P(e(\varphi), b) = 0}} \log |b| d\varphi.$$

The above construction gives us

$$\begin{aligned} \int_0^1 \sum_{\substack{0 \neq b \in B \\ P(e(\varphi), b) = 0}} \log |b| d\varphi &= \sum_{k=0}^{s-1} \int_{I_k} \sum_{\substack{0 \neq b \in B \\ P(e(\varphi), b) = 0}} \log |b| d\varphi \\ &= \sum_{k=0}^{s-1} \sum_{\substack{i \in \{1, \dots, n\} \\ \gamma_{ik}(I_k) \subset S^1 \times B}} \int_{I_k} \log |F_{ik}(e(\varphi))| d\varphi. \end{aligned}$$

We now have to show that

$$\int_{I_k} \gamma_{ik}^* C_2 = (-2\pi i) \int_{I_k} \log |F_{ik}(e(\varphi))| d\varphi.$$

Using $\partial(\log |t_i|) = \frac{1}{2} \frac{dt_i}{t_i}$ we get

$$C_2(\log |t_1|, \log |t_2|) = \frac{1}{2} \left(\log |t_1| \frac{dt_2}{t_2} - \log |t_2| \frac{dt_1}{t_1} - \log |t_1| \frac{d\bar{t}_2}{t_2} + \log |t_2| \frac{d\bar{t}_1}{t_1} \right).$$

According to the definition we have

$$\gamma_{ik}(\varphi) = (e(\varphi), F_{ik}(e(\varphi))).$$

Computing $\gamma_{ik}^* C_2$ one sees immediately (notice that $\log |e(\varphi)| = 0$)

$$\begin{aligned} \gamma_{ik}^* C_2 &= \frac{1}{2} (-\log |F_{ik}(e(\varphi))| e(-\varphi) (2\pi i) e(\varphi) d\varphi \\ &\quad + \log |F_{ik}(e(\varphi))| e(\varphi) (-2\pi i) e(-\varphi) d\varphi) \\ &= (-2\pi i) \log |F_{ik}(e(\varphi))| d\varphi. \end{aligned}$$

□

Corollary 1.3 *Using the above notation we get*

$$m(P) - m(P^*) = \frac{1}{2\pi i} \sum_{k=0}^{s-1} \sum_{\substack{i \in \{1, \dots, n\} \\ \gamma_{ik}(I_k) \subset S^1 \times B}} \int_{I_k} \gamma_{ik}^* C_2.$$

Proof: Obvious. □

Let us now fix some notations. Let $K = \mathbb{C}$ or \mathbb{R} . For a variety X over $K = \mathbb{R}$ we get an antiholomorphic involution F_∞ on $X(\mathbb{C})$. For a complex

\mathbb{C} -valued form η on $X(\mathbb{C})$ set $\overline{F}_\infty^* \eta = \overline{F_\infty^* \eta}$.

For any variety X/K and any subgroup $\Lambda \subset \mathbb{C}$ which in case $K = \mathbb{R}$ should in addition satisfy $\overline{\Lambda} = \Lambda$ we set

$$\begin{aligned} H^n(X/\mathbb{C}, \Lambda) &:= H_{\text{sing}}^n(X(\mathbb{C}), \Lambda) & \text{and} \\ H^n(X/\mathbb{R}, \Lambda) &:= H_{\text{sing}}^n(X(\mathbb{C}), \Lambda)^+ \end{aligned}$$

where the superscript $+$ denotes taking invariants under the action of \overline{F}_∞^* . A similar definition applies to homology and to relative situations.

Set $\Lambda(n) := (2\pi i)^n \Lambda$. We also need the natural pairing

$$\langle \cdot, \cdot \rangle : H^n(X/K, \mathbb{R}(n)) \times H_n(X/K, \mathbb{R}(-n)) \rightarrow \mathbb{R}$$

and again similar for relative situations. As a last ingredient we want to mention the fact that for $n > \dim X$ we have the equation $H_{\mathcal{D}}^{i+1}(X/K, \mathbb{R}(n)) = H^i(X/K, \mathbb{R}(n-1))$.

Let us now return to our main discussion. Connecting the paths γ_{ik} in an appropriate way using each path just one time and reparametrizing the resulting path we get closed paths $\chi_\mu : [0, 1] \rightarrow Z^*(P)^{\text{reg}}$ ($\mu = 0, \dots, \mu_0$) and paths with boundary $\psi_\nu : [0, 1] \rightarrow Z^*(P)^{\text{reg}}$ ($\nu = 0, \dots, \nu_0$) satisfying the following properties:

1. The boundary points of the ψ_ν are exactly those points where the number of paths γ_{ik} running into the point is not equal to the number of paths γ_{ik} running out of that point. Denote the set of all boundary points of paths ψ_ν by R_P . One has $R_P \subset T^2$.
2. We have

$$A = \bigcup_{\mu=0}^{\mu_0} \chi_\mu([0, 1]) \cup \bigcup_{\nu=0}^{\nu_0} \psi_\nu([0, 1]).$$

The paths χ_μ and ψ_ν give us classes $[\chi_\mu] \in H_1(Z^*(P)^{\text{reg}}/\mathbb{C}; \mathbb{Z})$ and $[\psi_\nu] \in H_1((Z^*(P)^{\text{reg}}, R_P)/\mathbb{C}; \mathbb{Z})$. Considering the $[\chi_\mu]$ also as elements of $H_1((Z^*(P)^{\text{reg}}, R_P)/\mathbb{C}; \mathbb{Z})$ we set

$$[A] := \sum_{\mu=0}^{\mu_0} [\chi_\mu] + \sum_{\nu=0}^{\nu_0} [\psi_\nu].$$

Now note that the restriction of the 1-form C_2 to $Z^*(P)^{\text{reg}}$ is closed therefore defining a cohomology class $[C_2] \in H^1(Z^*(P)^{\text{reg}}, \mathbb{R}(1))$. Since the restriction of C_2 to $R_P \subset T^2$ is zero we may also view it as defining a relative cohomology class $[C_2] \in H^1((Z^*(P)^{\text{reg}}, R_P)/\mathbb{C}; \mathbb{R}(1))$.

Using de Rham theorem it is not hard to show the following claim:

Theorem 1.4 *Let P be as above. There is a class*

$$[A] \otimes (2\pi i)^{-1} \in H_1((Z^*(P)^{reg}, R_P)/\mathbb{C}; \mathbb{Z}(-1)),$$

satisfying

$$(4) \quad m(P) - m(P^*) = \langle [C_2], [A] \otimes (2\pi i)^{-1} \rangle.$$

Remark 1.5 For polynomials $P \in \mathbb{C}[t_1, \dots, t_n]$ such that $Z^*(P)$ is smooth and does not intersect T^n a cohomological generalization of formula (4) was given in [4] Proposition 2.2.

Corollary 1.6 *Let $P \in \mathbb{Q}[t_1, t_2]$ be as above and assume $R_P = \emptyset$. Then we have*

$$[A] \otimes (2\pi i)^{-1} \in H_1(Z^*(P)^{reg}/\mathbb{R}, \mathbb{Z}(-1))$$

and

$$m(P) - m(P^*) = \langle r_{\mathcal{D}}(\{t_1, t_2\}), [A] \otimes (2\pi i)^{-1} \rangle,$$

where $\{t_1, t_2\} \in H_{\mathcal{M}}^2(Z^(P)^{reg}, \mathbb{Q}(2))$ and*

$$r_{\mathcal{D}} : H_{\mathcal{M}}^2(Z^*(P)^{reg}, \mathbb{Q}(2)) \rightarrow H_{\mathcal{D}}^2(Z^*(P)^{reg}, \mathbb{R}(2))$$

denotes as usual the regulator.

Proof: We calculate

$$\begin{aligned} [C_2] &= [\log |t_1|] \cup [\log |t_2|] && \text{see 1.1} \\ &= r_{\mathcal{D}}(t_1) \cup r_{\mathcal{D}}(t_2) \\ &= r_{\mathcal{D}}(\{t_1, t_2\}) \end{aligned}$$

using the compatibility of the regulator with respect to cup products and the fact that $r_{\mathcal{D}}(t_i) = \log |t_i|$. \square

Our assumptions imposed on the polynomial P at the beginning of the chapter are very restrictive. The following lemma allows us to weaken those conditions. But before doing so we need another notation. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z})$ and define

$$\begin{aligned} \phi_A : (\mathbb{C}^*)^2 &\rightarrow (\mathbb{C}^*)^2 \\ (t_1, t_2) &\mapsto (t_1^a t_2^c, t_1^b t_2^d). \end{aligned}$$

Lemma 1.7 *Let $0 \neq P(t_1, t_2) \in \mathbb{C}[t_1, t_2]$ written as in (3) and assume $P^*(t_1) = a_0(t_1)$. Let $Z^*(P)^{sing} = \{(z_1^{(i)}, z_2^{(i)}) \mid i = 1, \dots, r\}$ be the finite set of singularities of $Z^*(P)$. Assume $Z^*(P)^{sing} \cap T^2 = \emptyset$. Then there exists an $A \in GL_2(\mathbb{Z})$ so that $Q(t_1, t_2) := (t_1 t_2)^{\deg(a_0)} \cdot (\phi_A^* P)(t_1, t_2)$ satisfies the following conditions:*

1. $m(Q) = m(P)$.
2. $Q(t_1, t_2) \in \mathbb{C}[t_1, t_2]$.
3. $Q^*(t_1) = Q(t_1, 0)$ and $Q^*(t_1)$ is equal to the leading coefficient of P^* .
4. $Z^*(Q)^{sing} \cap (S^1 \times B) = (\phi_A)^{-1}(Z^*(P)^{sing}) \cap (S^1 \times B) = \emptyset$.
5. If P is irreducible, so is Q .
6. If P is reciprocal, so is Q .

Proof: Everything is obvious except of 4.: Choose $m_1 \in \mathbb{N}$ with $m_1 \geq \deg(a_i)$ for all $i \geq 1$. Let $\lambda_j^{(i)} := \log |z_j^{(i)}|$. Choose in addition $m_2 \in \mathbb{N}$ such that

$$(5) \quad (m_2 + 1)\lambda_1^{(i)} + \lambda_2^{(i)} \neq 0$$

for all $i = 1, \dots, r$. Let $m := \max\{m_1, m_2\}$ and

$$A := \begin{pmatrix} -1 & m \\ -1 & m+1 \end{pmatrix} \in GL_2(\mathbb{Z}).$$

It is now easily seen that 4. holds for A defined as above. \square

Remark 1.8 Roughly spoken the last lemma says that by changing to the polynomial Q we can get rid of singularities in $S^1 \times B \cap Z(P)$ as long as we assume $Z^*(P)^{sing} \cap T^2 = \emptyset$. For two variable polynomials this means that we no longer need the condition $A \subset Z^*(P)^{reg}$ which origins in “Assumptions 3.2” from [11].

Corollary 1.9 *Let $P \in \mathbb{Q}[t_1, t_2]$ be absolutely irreducible and reciprocal. In addition assume $Z^*(P)^{sing} \cap T^2 = \emptyset$. Denote by λ the leading coefficient of $P^*(t_1)$. Then there exists a class $[A]_P \otimes (2\pi i)^{-1} \in H_1(Z^*(P)^{reg}, \mathbb{Z}(-1))$, with*

$$m(P) = \log |\lambda| \pm \langle r_{\mathcal{D}}(\{t_1, t_2\}), [A]_P \otimes (2\pi i)^{-1} \rangle.$$

Proof: We want to use 1.7 to reduce to the situation of 1.6. To do so we still have to show that $R_Q = \emptyset$. 1.7 3. shows us that Q does not vanish at points of the form $(\cdot, 0)$. Take $(e(\varphi), re(\psi)) \in S^1 \times \mathbb{C}^*$ with $Q(e(\varphi), re(\psi)) = 0$. Due to 1.7 6. Q is reciprocal and we have

$$Q(e(-\varphi), \frac{1}{r}e(-\psi)) = 0.$$

Applying complex conjugation gives us

$$Q(e(\varphi), \frac{1}{r}e(\psi)) = 0.$$

Suppose we have a path in $Z^*(Q)$ coming from $S^1 \times B$ intersecting T^2 in a point and then leaving $S^1 \times B$. The above calculation then shows us that another path in $Z^*(Q)$ comes from the outside of $S^1 \times B$ intersects T^2 in the same point as above and runs into $S^1 \times B$. If we have two pathes both running in T^2 we can discard one (because it doesn't contribute to the integral we are considering). From these observations we get $R_Q = \emptyset$. Let $[A]$ be the class from 1.4 built with respect to our polynomial Q . Using the isomorphism $\phi_A : Z^*(Q)^{\text{reg}} \rightarrow Z^*(P)^{\text{reg}}$ from 1.7 we set $[A]_P := (\phi_A)_*[A]$. Applying 1.6 and 1.7 we conclude the proof. \square

Remark 1.10 1. Starting from his numerical experiments in [7] Boyd has conjectured that for reciprocal polynomials with zero locus of genus 1 one has always formulas analogous to (2). Our last result explains this in some way: the fact that the polynomial in question is reciprocal allows us to deal with absolute homology or cohomology classes rather than with relative ones.

2. The assumption $Z^*(P)^{\text{sing}} \cap T^2 = \emptyset$ seems to be crucial as the following example shows: Set

$$P(t_1, t_2) = (t_1^2 + t_1 + 1)t_2^2 + (t_1^4 - t_1^3 - 6t_1^2 - t_1 + 1)t_2 + t_1^2(t_1^2 + t_1 + 1)$$

$Z(P)$ is of genus 1 and $(S^1 \times B) \cap Z(P)$ is a closed path on which the singular points $(-1, 1), (1, 1)$ lie. Boyd gets numerically the unexpected formula

$$m(P) = *L'(\chi_1, -1) + *L'(\chi_2, -1)$$

where χ_1 and χ_2 are two Dirichlet characters. Seemingly one has to build the normalization of the curve $Z(P)$ first.

Corollary 1.11 *Let $P \in \mathbb{Q}[t_1, t_2]$ be absolutely irreducible and assume that $Z(P) \cap T^2 = \emptyset$. Denote by λ the leading coefficient of $P^*(t_1)$. There exists a class $[A]_P \otimes (2\pi i)^{-1} \in H_1(Z^*(P)^{\text{reg}}, \mathbb{Z}(-1))$, such that*

$$m(P) = \log |\lambda| \pm \langle r_{\mathcal{D}}(\{t_1, t_2\}), [A]_P \otimes (2\pi i)^{-1} \rangle.$$

Proof: Again we use 1.6 and 1.7. □

2 Boundary maps in K -theory

According to our general policy we want to use Beilinson's conjectures and theorems 1.6, 1.9 and 1.11 to produce formulas like (2). Since Beilinson's conjectures deal with projective, smooth varieties over \mathbb{Q} we need to know that our symbol $\{t_1, t_2\}$ already lies in the motivic cohomology of the projective, smooth model of our initial curve, i.e. we need to know that our symbol vanishes under the tame symbol.

We have to fix some notations. Let

$$P(t_1, t_2) = \sum_{k_1, k_2} \alpha_{k_1, k_2} t_1^{k_1} t_2^{k_2} \in \mathbb{Q}[t_1, t_2]$$

an absolute irreducible polynomial. Denote by $Z(P)$ the algebraic variety over \mathbb{Q} defined by the equation $P = 0$. Let C denote the normalization of the projective closure of $Z(P)$. Consider t_1, t_2 as rational functions on C . Set $S := \text{supp}(\text{div}(t_1)) \cup \text{supp}(\text{div}(t_2))$ and $U := C - S$.

Define the Newton polygon $\mathcal{N}(P)$ of our polynomial P to be the convex hull of the set $\{(k_1, k_2) \mid \alpha_{k_1, k_2} \neq 0\}$ in \mathbb{R}^2 . For a side F of $\mathcal{N}(P)$ we parametrize the points of $F \cap \mathbb{Z}^2$ clockwise in such a manner that $(k_1^0, k_2^0), \dots, (k_1^l, k_2^l)$ are the consecutive lattice points of F . One can attach to every side F of $\mathcal{N}(P)$ a one-variable polynomial

$$P_F(t) := \sum_{i=0}^l \alpha_{k_1^i, k_2^i} t^i \in \mathbb{Q}[t].$$

Boyd calls a polynomial P tempered if all P_F for all sides F of $\mathcal{N}(P)$ have only roots of unity as zeroes.

Let us now return to K -theory. Obviously one has $\{t_1, t_2\} \in H_{\mathcal{M}}^2(U, \mathbb{Q}(2))$. As mentioned above we want to know under which assumptions $\{t_1, t_2\} \in H_{\mathcal{M}}^2(C, \mathbb{Q}(2))$ holds. The following theorem gives the answer:

Theorem 2.1 *With notations as above the following two conditions are equivalent*

- (1) $\{t_1, t_2\} \in H_{\mathcal{M}}^2(C, \mathbb{Q}(2))$.
- (2) P is tempered.

Proof: The general case is due to Rodriguez-Villegas (see [15] chapter 8). For the special form $P(t_1, t_2) = A(t_1)t_2^2 + B(t_1)t_2 + C(t_1)$ of polynomials considered by Boyd the proof is an easy but tedious calculation. At first one has to calculate the divisors of t_1 and t_2 as rational functions on C . After doing so one can determine the tame symbol

$$K_2(\mathbb{Q}(C)) \otimes \mathbb{Q} \xrightarrow{\partial = \coprod_p \partial_p} \coprod_{p \in C(\mathbb{Q})} \mathbb{Q}(p)^* \otimes \mathbb{Q}.$$

where

$$\partial_p(\{f, g\}) = \left[(-1)^{\text{ord}_p(f) \text{ord}_p(g)} \frac{f^{\text{ord}_p(g)}}{g^{\text{ord}_p(f)}} \right] (p) \otimes 1.$$

It shows up that while p runs over $p \in \text{supp}(\text{div}(t_1)) \cup \text{supp}(\text{div}(t_2))$ there always exists a zero ζ_p of a polynomial P_F for a side F of $\mathcal{N}(P)$ such that

$$\partial_p(\{t_1, t_2\}) = \zeta_p \otimes 1.$$

This takes care of the implication (2) \Rightarrow (1). One also notes that for every side F of $\mathcal{N}(P)$ and every zero ζ of the polynomial P_F attached to the side there is a $p_\zeta \in \text{supp}(\text{div}(t_1)) \cup \text{supp}(\text{div}(t_2))$ such that

$$\partial_{p_\zeta}(\{t_1, t_2\}) = \zeta \otimes 1.$$

This gives us the implication (1) \Rightarrow (2). □

Let us now assume that C is an elliptic curve over \mathbb{Q} , i. e. is of genus 1 and has got a \mathbb{Q} -rational point. Beilinson's conjectures deal with $H_{\mathcal{M}}^2(C, \mathbb{Q}(2))_{\mathbb{Z}} \subset H_{\mathcal{M}}^2(C, \mathbb{Q}(2))$. So even if our symbol is already an element of $H_{\mathcal{M}}^2(C, \mathbb{Q}(2))$ it has to overcome another obstruction, the so called integral obstruction δ (\mathcal{C}/\mathbb{Z} denotes the minimal regular model of C):

$$0 \rightarrow H_{\mathcal{M}}^2(C, \mathbb{Q}(2))_{\mathbb{Z}} \rightarrow H_{\mathcal{M}}^2(C, \mathbb{Q}(2)) \xrightarrow{\delta = \coprod_p \delta_p} \coprod_p K'_1(\mathcal{C}_p) \otimes \mathbb{Q} \rightarrow \dots$$

The following theorem gives us an example for a whole family of curves where the integral obstruction of a certain symbol vanishes (enabling us to produce a formula like (2)).

Theorem 2.2 *Take the following family of polynomials from $\mathbb{Z}[t_1, t_2]$:*

$$P_k(t_1, t_2) := t_1 t_2^2 + (t_1^2 + k t_1 + 1) t_2 + t_1.$$

Assume $k \in \mathbb{Z} - \{0, \pm 4\}$. Then we have

1. *The zero locus $Z(P_k)$ is birationally equivalent to an elliptic curve C_k over \mathbb{Q} .*
2. *Assuming Beilinson's conjectures for elliptic curves we get*

$$m(P_k) = *L'(C_k, 0)$$

Proof: Assume $k \in \mathbb{Z} - \{0, \pm 4\}$. Let $C = C_k$ be the elliptic curve defined by the Weierstrass equation

$$(6) \quad y^2 + kxy + ky = x^3 + x^2.$$

The map

$$\begin{aligned} Z(P_k) &\rightarrow C \\ (t_1, t_2) &\mapsto (k(t_1 + t_2)^{-1}, -k(t_1 + t_2)^{-2} t_1(t_1 + t_2 + k)) \end{aligned}$$

establishes a birational equivalence between the two curves thereby taking care of our first claim. Now using 1.9 we show

$$(7) \quad m(P_k) = \pm \langle r_{\mathcal{D}}\{t_1, t_2\}, \gamma_k \rangle$$

with $\{t_1, t_2\} \in H_{\mathcal{M}}^2(Z^*(P_k)^{\text{reg}}, \mathbb{Q}(2))$ and $\gamma_k \in H_1(Z^*(P_k)^{\text{reg}}, \mathbb{Z}(-1))$. Theorem 2.1 gives us

$$\{t_1, t_2\} \in H_{\mathcal{M}}^2(C, \mathbb{Q}(2)).$$

Now we have to calculate the integral obstruction of the symbol $\{t_1, t_2\}$. If the reduction at p is not split multiplicative we have $K'_1(\mathcal{C}_p) \otimes \mathbb{Q} = 0$. Therefore we confine ourselves to the case of split multiplicative reduction at p . Then \mathcal{C}_p is a Néron N -gon. For the divisors

$$\text{div}(t_1) = \sum_i a_i(X_i) \text{ and } \text{div}(t_2) = \sum_j b_j(Y_j)$$

set

$$d_{t_1}(\nu) = \sum_i a_i d_{X_i}(\nu),$$

where

$$d_{X_i}(\nu) = \begin{cases} 1 & \text{if } X_i \text{ reduces to the } \nu\text{-th side of the } N\text{-gon} \\ 0 & \text{else.} \end{cases}$$

Using this notation we have the following formula which is due to Schapacher and Scholl (see [16] chapter 3):

$$(8) \quad \delta_p(\{t_1, t_2\}) = \pm \frac{1}{3N} \sum_{\substack{\mu \in \mathbb{Z} \\ \nu \in \mathbb{Z}}} d_{t_1}(\mu) d_{t_2}(\nu + \mu) B_3 \left(\left\langle \frac{\nu}{N} \right\rangle \right).$$

Here $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ (the third Bernoulli polynomial) and $\langle \frac{\nu}{N} \rangle \equiv \frac{\nu}{N} \bmod \mathbb{Z}$ subject to the condition $\langle \frac{\nu}{N} \rangle \in [0, 1[$.

Returning to our special situation let us remark that for the curve C we get $c_4 = k^4 - 16k^2 + 16$ and $\Delta = k^2(k-4)(k+4)$. Furthermore on C our divisors read as

$$\begin{aligned} \operatorname{div}(t_1) &= (O) + (Q) - (2Q) - (3Q) & \text{and} \\ \operatorname{div}(t_2) &= (O) - (Q) - (2Q) + (3Q) \end{aligned}$$

where O is the origin of C and $Q = (0, 0)$. Clearly we have $2Q = (-1, 0)$, $3Q = (0, -k)$ and $4Q = O$.

In computing the reduction of C and the four points $O, Q, 2Q, 3Q$ on it at a prime p let us first assume $p \geq 3$. Clearly the inequation $v_p(\Delta) > 0$ is then equivalent to having $p|k$, $p|(k-4)$ or $p|(k+4)$. Say $p|k$. Since $v_p(c_4) = 0$ our Weierstrass equation (6) is minimal and it has multiplicative reduction at p which in addition we assume to be split multiplicative. The reduced equation

$$\tilde{C} : y^2 = x^3 + x^2$$

has got the singular point $(0, 0)$. Now let $\tilde{C}_{\text{ns}}(\mathbb{F}_p)$ denote the set of non-singular points of $\tilde{C}(\mathbb{F}_p)$ i.e. in our setting $\tilde{C}_{\text{ns}}(\mathbb{F}_p) = \tilde{C}(\mathbb{F}_p) - \{(0, 0)\}$. Furthermore set $C_0(\mathbb{Q}_p) = \{P \in C(\mathbb{Q}_p) \mid \tilde{P} \in \tilde{C}_{\text{ns}}(\mathbb{F}_p)\}$. Clearly we have

$$(9) \quad \operatorname{ord}_{C(\mathbb{Q}_p)/C_0(\mathbb{Q}_p)}(mQ) = \begin{cases} 1 & \text{for } m = 0, 2 \\ 2 & \text{for } m = 1, 3. \end{cases}$$

Let us now consider the following well known fact on the the Néron model \mathcal{N}/\mathbb{Z}_p of the elliptic curve C/\mathbb{Q}_p : set $\tilde{\mathcal{N}} = \mathcal{N} \times_{\mathbb{Z}_p} \mathbb{F}_p$ and let $\tilde{\mathcal{N}}^0$ denote the component of the identity in the group variety $\tilde{\mathcal{N}}$. Then under the identification $\mathcal{N}(\mathbb{Z}_p) \cong C(\mathbb{Q}_p)$ we get

$$\tilde{\mathcal{N}}(\mathbb{F}_p)/\tilde{\mathcal{N}}^0(\mathbb{F}_p) \cong C(\mathbb{Q}_p)/C_0(\mathbb{Q}_p).$$

Using this fact and (9) we have

$$\begin{aligned} d_O(\nu) &= d_{2Q}(\nu) & \text{and} \\ d_Q(\nu) &= d_{3Q}(\nu) \end{aligned}$$

and hence

$$d_{t_1}(\nu) = 1 \cdot d_O(\nu) + 1 \cdot d_Q(\nu) - 1 \cdot d_{2Q}(\nu) - 1 \cdot d_{3Q}(\nu) = 0.$$

The cases $p \geq 3, p|(k-4)$ and $p \geq 3, p|(k+4)$ proceed in a very similar line and are therefore omitted. In the case $p = 2$ the reduction is additive for $0 < v_2(k) < 4$. For $v_2(k) \geq 4$ one changes to a minimal Weierstrass equation and concludes almost verbatim like above.

After all we get

$$\{t_1, t_2\} \in H_{\mathcal{M}}^2(C_k, \mathbb{Q}(2))_{\mathbb{Z}}.$$

A standard inequality from the theory of Mahler measures shows us that $m(P_k) \neq 0$ and therefore by (7) $\{t_1, t_2\} \neq 0$ and $\gamma_k \neq 0$. Now using Beilinson's conjectures for elliptic curves we get

$$m(P_k) = *L'(C_k, 0).$$

□

Remark 2.3 1. Rodriguez-Villegas has announced that he found theoretical arguments for the vanishing of the integral obstruction of certain symbols.

2. Boyd has also given several examples for which it is possible to prove a formula like (2) rigorously, i.e without assuming the validity of Beilinson's conjectures. In these examples the elliptic curves in consideration have got CM. This crucial fact allows one to apply methods from [12]. For the details see [5] chapter 5.5.

3 Curves of genus 2

In [7] Boyd has also computed lots of examples where curves of genus 2 occur. Set for example

$$P(t_1, t_2) := (t_1^2 + t_1 + 1)t_2^2 + t_1(t_1 + 1)t_2 + t_1(t_1^2 + t_1 + 1).$$

Let C again be the normalization of the projective closure of $Z(P)$. The curve C has genus 2. Its Jacobian $J(C)$ is reducible, i. e. it is isogenous to a product of two elliptic curves. Numerically it seems that

$$m(P) = *L'(E, 0),$$

where E is one of the above factors of the Jacobian. It is by no means clear why the Mahler measure “ignores” the other elliptic curve. In this chapter we exhibit the K -theoretical reasons for this behaviour.

First we have to fix notations. Let $P(t_1, t_2) := A(t_1)t_2^2 + B(t_1)t_2 + C(t_1) \in \mathbb{Z}[t_1, t_2]$ a tempered, reciprocal polynomial. Set $D(t_1) := B(t_1)^2 - 4A(t_1)C(t_1)$. Assume that $D(t_1) = (t_1 + 1)^{2r} \tilde{D}(t_1)$, where $r \in \mathbb{N}$ and $\tilde{D} \in \mathbb{Z}[t_1]$ is of degree 5 or 6 with non-vanishing discriminant. Furthermore let s be the unique natural number subject to the condition

$$P(t_1, t_2) = t_1^s t_2^2 P\left(\frac{1}{t_1}, \frac{1}{t_2}\right).$$

Assume finally that $s = 3 + r$. This is in some way a natural assumption because it follows easily from the above assumptions that we always have $s \geq 3 + r$.

One defines easily a birational equivalence from $Z(P)$ to the curve $Z(y^2 - \tilde{D}(x))$. Furthermore it can be shown that $t_1^6 \tilde{D}(\frac{1}{t_1}) = \tilde{D}(t_1)$. Using the transformation

$$\begin{aligned} x &= \frac{S+1}{S-1} \\ y &= \frac{T}{(S-1)^3} \end{aligned}$$

(see [8] p. 160) we can get our curve $Z(y^2 - \tilde{D}(x))$ birational equivalent to a curve with model $T^2 = Q(S^2)$, where $Q(z) := c_3 z^3 + c_2 z^2 + c_1 z + c_0 \in \mathbb{Q}[z]$ with non-vanishing discriminant and $c_0 c_3 \neq 0$. Let C be the normalization of the projective closure of $T^2 = Q(S^2)$. Define θ to be the symbol $\{t_1, t_2\}$ on $Z(P)$ transformed to our current model $T^2 = Q(S^2)$. Since P is tempered we have $\theta \in H_{\mathcal{M}}^2(C, \mathbb{Q}(2))$.

In this situation we use Theorem 14.1.1 from [8]. As in the proof of the theorem we define two elliptic curves $E_1 : w^2 = Q(z)$ and $E_2 : w^2 = z^3 Q(\frac{1}{z})$ and two Galois coverings $\varphi_1 : C \rightarrow E_1, (S, T) \mapsto (S^2, T)$ and $\varphi_2 : C \rightarrow E_2, (S, T) \mapsto (S^{-2}, TS^{-3})$. The Galois group $\text{Gal}(\mathbb{Q}(C)/\varphi_1^* \mathbb{Q}(E_1))$ is generated

by

$$\begin{aligned}\tau_1 : \mathbb{Q}(C) &\rightarrow \mathbb{Q}(C) \\ S &\mapsto -S \\ T &\mapsto T.\end{aligned}$$

One projection formula from motivic cohomology reads therefore $\varphi_1^* \circ \varphi_{1,*} = \text{id} + \tau_1$. Since we assume P to be tempered we get

$$\begin{aligned}\theta &= \left\{ \frac{S+1}{S-1}, \frac{\frac{(2S)^r T}{(S-1)^{3+r}} - B\left(\frac{S+1}{S-1}\right)}{2A\left(\frac{S+1}{S-1}\right)} \right\} \\ &= \left\{ \frac{S+1}{S-1}, \frac{\frac{(2S)^r T}{(S-1)^{3+r}} - B\left(\frac{S+1}{S-1}\right)}{2\alpha} \right\}\end{aligned}$$

where α denotes the leading coefficient of the polynomial A . In the above computation we have used the fact $\{x, x - \zeta\} = 0$ for ζ a root of unity. Now applying our assumptions P reciprocal and $s = 3 + r$ we compute

$$\begin{aligned}\tau_1 \theta &= \left\{ \frac{S-1}{S+1}, \frac{-\frac{(2S)^r T}{(S+1)^{3+r}} - B\left(\frac{S-1}{S+1}\right)}{2\alpha} \right\} \\ &= \left\{ \frac{S-1}{S+1}, \frac{-\frac{(2S)^r T}{(S+1)^{3+r}} - \left(\frac{S-1}{S+1}\right)^s B\left(\frac{S+1}{S-1}\right)}{2\alpha} \right\} \\ &= \left\{ \frac{S-1}{S+1}, \left(\frac{S-1}{S+1}\right)^s \frac{-\frac{(2S)^r T}{(S-1)^{3+r}} - B\left(\frac{S+1}{S-1}\right)}{2\alpha} \right\} \\ &= \left\{ \frac{S-1}{S+1}, \frac{\frac{(2S)^r T}{(S-1)^{3+r}} + B\left(\frac{S+1}{S-1}\right)}{-2\alpha} \right\}.\end{aligned}$$

Let us now denote by γ the leading coefficient of the polynomial C . Since P is tempered we clearly have $\alpha = \pm\gamma$. Further using the fact

$$\frac{\frac{(2S)^r T}{(S-1)^{3+r}} + B\left(\frac{S+1}{S-1}\right)}{-2C\left(\frac{S+1}{S-1}\right)} = \frac{2A\left(\frac{S+1}{S-1}\right)}{\frac{(2S)^r T}{(S-1)^{3+r}} - B\left(\frac{S+1}{S-1}\right)}$$

we proceed in doing our computation

$$\begin{aligned}
\tau_1 \theta &= \left\{ \frac{S-1}{S+1}, \frac{\frac{(2S)^r T}{(S-1)^{3+r}} + B\left(\frac{S+1}{S-1}\right)}{-2\gamma} \right\} \\
&= \left\{ \frac{S-1}{S+1}, \frac{\frac{(2S)^r T}{(S-1)^{3+r}} + B\left(\frac{S+1}{S-1}\right)}{-2C\left(\frac{S+1}{S-1}\right)} \right\} \\
&= \left\{ \frac{S-1}{S+1}, \frac{2A\left(\frac{S+1}{S-1}\right)}{\frac{(2S)^r T}{(S-1)^{3+r}} - B\left(\frac{S+1}{S-1}\right)} \right\} \\
&= \left\{ \frac{S+1}{S-1}, \frac{\frac{(2S)^r T}{(S-1)^{3+r}} - B\left(\frac{S+1}{S-1}\right)}{2A\left(\frac{S+1}{S-1}\right)} \right\} \\
&= \theta.
\end{aligned}$$

At last we get

$$\varphi_1^* \circ \varphi_{1,*} \theta = 2\theta.$$

An analogous computation can be done using the Galois covering φ_2 . The result is

$$\varphi_2^* \circ \varphi_{2,*} \theta = 0$$

and therefore $\varphi_{2,*} \theta = 0$.

The above discussion is the main ingredient in the following theorem.

Theorem 3.1 *Let the assumption of the above discussion apply. Furthermore*

1. *let P be absolute irreducible,*
2. *let the coefficients of P associated to the extremal points of $\mathcal{N}(P)$ have absolute value 1,*
3. *let $Z^*(P)^{sing} \cap T^2 = \emptyset$ and*
4. *let $m(P) \neq 0$.*

In addition assume

$$\varphi_{1,*} \theta \in H_{\mathcal{M}}^2(E_1, \mathbb{Q}(2))_{\mathbb{Z}}$$

and the validity of the Beilinson conjectures for elliptic curves. One has

$$m(P) = *L'(E_1, 0).$$

Proof: Using 1.9 and transforming everything into the model C we get

$$m(P) = \pm \langle r_{\mathcal{D}}(\theta), \gamma \rangle$$

where $\gamma \in H_1(C, \mathbb{Z}(-1))$. We conclude

$$\begin{aligned} m(P) &= \pm \langle r_{\mathcal{D}}(\theta), \gamma \rangle \\ &= \pm \frac{1}{2} \langle r_{\mathcal{D}}(2\theta), \gamma \rangle \\ &= \pm \frac{1}{2} \langle r_{\mathcal{D}}((\varphi_1^* \circ \varphi_{1,*})\theta), \gamma \rangle \\ &= \pm \frac{1}{2} \langle \varphi_1^* r_{\mathcal{D}}(\varphi_{1,*}\theta), \gamma \rangle \\ &= \pm \frac{1}{2} \langle r_{\mathcal{D}}(\varphi_{1,*}\theta), \varphi_{1,*}\gamma \rangle. \end{aligned}$$

Since $m(P) \neq 0$ we have $\varphi_{1,*}\gamma \neq 0$ and $\varphi_{1,*}\theta \neq 0$. Using Beilinson conjectures we finally get

$$m(P) = *L'(E_1, 0).$$

□

4 Formulas of mixed type

Another interesting example of Boyd is given by

$$(10) \quad P(t_1, t_2) := (t_1^2 + 1)^2 t_2^2 + 2t_1 t_2 + 1.$$

Numerically evidence suggests

$$(11) \quad m(P) = *L'(E, 0) + *L'(\chi, -1)$$

to be true where E is an elliptic curve over \mathbb{Q} (defined as usual) and χ is the non-trivial Dirichlet character of $\mathbb{Z}/3\mathbb{Z}$.

Using the notation of chapter 1 we have

$$R_P = \{(\zeta_3, -\zeta_3^{-1}), (\zeta_3^{-1}, -\zeta_3), (\zeta_6, -\zeta_6^{-1}), (\zeta_6^{-1}, -\zeta_6)\}$$

where $\zeta_3 = \exp(\frac{2\pi i}{3})$ and $\zeta_6 = \exp(\frac{\pi i}{3})$. Theorem 1.4 gives us

$$(12) \quad m(P) = \langle [C_2], [A] \otimes (2\pi i)^{-1} \rangle$$

for a certain class $[A] \otimes (2\pi i)^{-1} \in H_1((Z^*(P)^{\text{reg}}, R_P)/\mathbb{R}; \mathbb{Z}(-1))$.

Let E denote as usual the non-singular projective model of $Z(P)$. This is an elliptic curve defined over \mathbb{Q} . Consider $R := R_P$ and $Z^*(P)^{\text{reg}}$ as subvarieties of E .

Set $R_{/\mathbb{Q}} = \text{Spec } \mathbb{Q}(\mu_{12})$. We can view $R_{/\mathbb{Q}}$ as subscheme of $E_{/\mathbb{Q}}$ in such a way that the points of $R_{/\mathbb{Q}}(\bar{\mathbb{Q}})$ correspond to the points of R in $E(\bar{\mathbb{Q}})$. Therefore we denote $R_{/\mathbb{Q}}$ also by R .

Consider t_1, t_2 as rational functions on E . Clearly we have $\{t_1, t_2\} \in H_{\mathcal{M}}^2(E, \mathbb{Q}(2))$. Since $t_i(Q)$ is a root of unity for $i \in \{1, 2\}$ and for every $Q \in R$ we can also view $\{t_1, t_2\}$ as an element of $H_{\mathcal{M}}^2(E, R; \mathbb{Q}(2))$. We have $r_{\mathcal{D}}(\{f, g\}) = [C_2]$ even in the relative situation.

In what follows set

$$\Gamma := [C_2] \in H_{\mathcal{D}}^2((E, R)_{/\mathbb{R}}, \mathbb{R}(2)) = H^1((E, R)_{/\mathbb{R}}, \mathbb{R}(1))$$

and view $\gamma := [A] \otimes (2\pi i)^{-1}$ as an element of $H_1((E, R)_{/\mathbb{R}}; \mathbb{Z}(-1))$. With this notation (12) reads

$$m(P) = \pm \langle \Gamma, \gamma \rangle.$$

One has the following birational map on $Z(P)$:

$$\begin{aligned} \sigma : Z(P) &\rightarrow Z(P) \\ (t_1, t_2) &\mapsto \left(t_1, \frac{t_2}{-1 - 2t_1 t_2} \right). \end{aligned}$$

This can be extended to an involution on E , which is defined over \mathbb{Q} . It is easy to see that $\sigma|_R = \text{id}$ and that for $[C_2] \in H_{\mathcal{D}}^2(E_{/\mathbb{R}}, \mathbb{R}(2)) = H^1(E_{/\mathbb{R}}, \mathbb{R}(1))$

$$\sigma^*[C_2] = -[C_2].$$

From relative, long exact sequences of algebraic topology we get for cohomology

$$\dots \rightarrow H^0(R_{/\mathbb{R}}, \mathbb{R}(1)) \xrightarrow{\delta^*} H^1((E, R)_{/\mathbb{R}}; \mathbb{R}(1)) \xrightarrow{j^*} H^1(E_{/\mathbb{R}}, \mathbb{R}(1)) \rightarrow 0.$$

and for homology

$$0 \rightarrow H_1(E_{/\mathbb{R}}, \mathbb{R}(-1)) \xrightarrow{j_*} H_1((E, R)_{/\mathbb{R}}; \mathbb{R}(-1)) \xrightarrow{\delta_*} H_0(R_{/\mathbb{R}}, \mathbb{R}(-1)) \rightarrow \dots$$

Here $j : (E, \emptyset) \rightarrow (E, R)$ denotes the inclusion and δ_* (respectively δ^*) the boundary operator.

Our involution σ gives us an $\langle \sigma \rangle$ -operation on all of the above groups which

makes all occuring homomorphisms and the several pairings $\langle \cdot, \cdot \rangle$ equivariant. Forcing our sequences to be short exact and choosing $\langle \sigma \rangle$ -equivariant homomorphisms s and t subject to the conditions $j^* \circ s = \text{id}$ and $\delta_* \circ t = \text{id}$ we have the two decompositions

$$\begin{aligned}\Gamma &= \delta^* \Gamma^0 + s \Gamma^1 & \text{and} \\ \gamma &= j_* \gamma_1 + t \gamma_0\end{aligned}$$

where

$$\begin{aligned}\Gamma^0 &\in H^0(R/\mathbb{R}, \mathbb{R}(1)), \\ \Gamma^1 &\in H^1(E/\mathbb{R}, \mathbb{R}(1)), \\ \gamma_0 &\in H_0(R/\mathbb{R}, \mathbb{Q}(-1)) & \text{and} \\ \gamma_1 &\in H_1(E/\mathbb{R}, \mathbb{Q}(-1)).\end{aligned}$$

Clearly we have $\Gamma^1 = [C_2] \in H^1(E/\mathbb{R}, \mathbb{R}(1))$ and therefore $\sigma \Gamma^1 = -\Gamma^1$. Using this and the fact that σ operates trivially on $H_0(R/\mathbb{R}, \mathbb{Q}(-1))$ we get

$$m(P) = \pm \underbrace{\langle \Gamma^1, \gamma_1 \rangle}_I \pm \underbrace{\langle \Gamma^0, \gamma_0 \rangle}_{II}.$$

Our usual reasoning shows us (modulo Beilinson's conjectures) that

$$\langle \Gamma^1, \gamma_1 \rangle = *L'(E, 0)$$

which takes care of term I. To give a proper meaning to term II is much harder. It turns out that we are not totally free in choosing a splitting of the above cohomology sequence.

For every two geometric points of R the difference is a torsion point of the elliptic curve E . It should be exact this property (as we will indicate in the next chapter) that allows us to choose splittings s which make the whole following diagram commute:

(13)

$$\begin{array}{ccccccc} H_{\mathcal{M}}^1(R, \mathbb{Q}(2)) & \xrightarrow{\delta^*} & H_{\mathcal{M}}^2(E, R; \mathbb{Q}(2)) & \xleftarrow[j^*]{s} & H_{\mathcal{M}}^2(E, \mathbb{Q}(2)) & \rightarrow & 0 \\ \downarrow r_{\mathcal{D}} & & \downarrow r_{\mathcal{D}} & & \downarrow r_{\mathcal{D}} & & \\ H_{\mathcal{D}}^1(R/\mathbb{R}, \mathbb{R}(2)) & \xrightarrow{\delta^*} & H_{\mathcal{D}}^2((E, R)/\mathbb{R}; \mathbb{R}(2)) & \xleftarrow[j^*]{s} & H_{\mathcal{D}}^2(E/\mathbb{R}, \mathbb{R}(2)) & \rightarrow & 0 \\ \parallel & & \parallel & & \parallel & & \\ H^0(R/\mathbb{R}, \mathbb{R}(1)) & \xrightarrow{\delta^*} & H^1((E, R)/\mathbb{R}; \mathbb{R}(1)) & \xleftarrow[j^*]{s} & H^1(E/\mathbb{R}, \mathbb{R}(1)) & \rightarrow & 0. \end{array}$$

The element $B := \{f, g\} \in H_{\mathcal{M}}^2(E, R; \mathbb{Q}(2))$ decomposes as follows

$$B = \delta^* B^0 + sB^1$$

where

$$\begin{aligned} B^0 &\in H_{\mathcal{M}}^1(R, \mathbb{Q}(2)) & \text{and} \\ B^1 &\in H_{\mathcal{M}}^2(E, \mathbb{Q}(2)). \end{aligned}$$

Evaluating $r_{\mathcal{D}}$ at B and comparing to Γ gives

$$\delta^* \Gamma^0 + s\Gamma^1 = (r_{\mathcal{D}} \circ \delta^*)B^0 + (r_{\mathcal{D}} \circ s)B^1.$$

Using $r_{\mathcal{D}} \circ s = s \circ r_{\mathcal{D}}$ and $r_{\mathcal{D}} B^1 = \Gamma^1$ we get

$$r_{\mathcal{D}} B^0 - \Gamma^0 \in \ker \delta^*$$

which in turn amounts to

$$\langle \Gamma^0, \gamma_0 \rangle = \langle r_{\mathcal{D}} B^0, \gamma_0 \rangle.$$

Using Beilinson's theorem on special values of Dirichlet L-functions and especially his explicit description of the regulator map

$$r_{\mathcal{D}} : H_{\mathcal{M}}^1(R, \mathbb{Q}(2)) \rightarrow H_{\mathcal{D}}^1(R/\mathbb{R}, \mathbb{R}(2))$$

(see [2] or [13]) we can interpret $\langle r_{\mathcal{D}} B^0, \gamma_0 \rangle$ in the way we intended to do. In our case Beilinson's theorem says that there is a map

$$\varepsilon_2 : \mu_{12} - \{1\} \rightarrow H_{\mathcal{M}}^1(R, \mathbb{Q}(2)),$$

such that

$$H_{\mathcal{M}}^1(R, \mathbb{Q}(2)) = \mathbb{Q} \cdot (\varepsilon_2(\xi) - \varepsilon_2(\xi^{-1})) \oplus \mathbb{Q} \cdot (\varepsilon_2(\xi^5) - \varepsilon_2(\xi^{-5})).$$

for $\xi = e^{\pi i/6}$. Now let ψ be the non-trivial Dirichlet character of $\mathbb{Z}/4\mathbb{Z}$ and let χ be as above. Beilinson's theorem further tells us

$$\begin{aligned} r_{\mathcal{D}}(\varepsilon_2(\xi) - \varepsilon_2(\xi^{-1})) &= (q_1 L'(\psi, -1) + q_2 L'(\chi, -1))\eta \\ &\quad + (q_1 L'(\psi, -1) - q_2 L'(\chi, -1))\eta' \end{aligned}$$

and

$$\begin{aligned} r_{\mathcal{D}}(\varepsilon_2(\xi^5) - \varepsilon_2(\xi^{-5})) &= (q_1 L'(\psi, -1) - q_2 L'(\chi, -1))\eta \\ &\quad + (q_1 L'(\psi, -1) + q_2 L'(\chi, -1))\eta' \end{aligned}$$

for $q_1, q_2 \in \mathbb{Q}^*$. Here

$$\eta = \begin{pmatrix} 2\pi i \\ 0 \\ -2\pi i \\ 0 \end{pmatrix} \quad \text{and} \quad \eta' = \begin{pmatrix} 0 \\ 2\pi i \\ 0 \\ -2\pi i \end{pmatrix}$$

is a \mathbb{Q} -basis of $H_{\mathcal{D}}^1(R/\mathbb{R}, \mathbb{R}(2)) = [(\mathbb{C}/\mathbb{R}(2))^{R(\mathbb{C})}]^+$. Therefore it shows up that we have

$$(14) \quad \langle r_{\mathcal{D}} B^0, \gamma_0 \rangle = \kappa_1 L'(\chi, -1) + \kappa_2 L'(\psi, -1)$$

where $\kappa_1, \kappa_2 \in \mathbb{Q}$. Together with term I we get

$$m(P) = *L'(E, 0) + \kappa_1 L'(\chi, -1) + \kappa_2 L'(\psi, -1).$$

Unfortunately this falls short of “proving” (11).

5 A general philosophy

In this last chapter let us first have a look at another interesting example due to Boyd. Let

$$(15) \quad P(t_1, t_2) := t_1^2 t_2^2 + t_1 + t_2 + 1.$$

The zero locus $Z(P)$ is birationally equivalent to the elliptic curve E defined by

$$y^2 + y = x^3 - x^2.$$

An easy calculation gives

$$Z(P) \cap T^2 = \{(-\zeta, \zeta) | \zeta^4 = -1\} \cup \{(-1, -1)\}.$$

Denote this set by R and consider it as a subvariety of E . Boyd has calculated $m(P)$ numerically using a precision of 25 decimal places. He gave this value together with the numerical values of $L'(E, 0)$ and $L'(\chi, -1)$ for some Dirichlet characters of conductor 8 as an input to a linear dependence finder (like for example *lindep* in the package *Pari*; see [1]). An intensive search using this method failed to produce a formula like (2) or (11).

Applying 1.4 to the polynomial $P(t_1 t_2, t_2) = t_1^2 t_2^4 + t_1 t_2 + t_2 + 1$ we easily see that

$$m(P) = \pm \langle r_{\mathcal{D}}(\Psi), \Phi \rangle$$

for certain elements $\Psi \in H_{\mathcal{M}}^2(E, R; \mathbb{Q}(2))$ and $\Phi \in H_1(E, R; \mathbb{Q}(-1))$. The question that arises from these considerations is: what is the basic difference between examples like (10) and (15)?

Denote by P_1, \dots, P_5 the five geometric points of R (again considered as points on E). Using *Pari* the author has calculated the multiples $[m](P_i - P_j)$ on E for $i \neq j$ and $m \leq 1000$. These calculations give strong evidence that the $P_i - P_j$ for $i \neq j$ are no torsion points on E at all.

As is easily seen the geometric points of the boundary R of chapter 4 have this property of every difference of points being a torsion point on the elliptic curve. This should answer our above question since in what follows we will give a heuristical argument why the mentioned property is crucial in finding a splitting in K -theory like (13).

Let K/\mathbb{Q} be a finite Galois extension subject to the condition that all geometric points of R are K -rational. Denote by \mathcal{MM}_K the (not yet constructed) category of mixed motives over K . Set $U := E - R$. By base extension and by applying the functor H^* we get motives $H^*(E_K), H^*(U_K)$ and $H^*(R_K)$. Take a look at the exact sequence

$$0 \rightarrow H^1(E_K)(1) \rightarrow H^1(U_K)(1) \rightarrow H^2(E_K, U_K)(1) \xrightarrow{j^*} H^2(E_K)(1)$$

in \mathcal{MM}_K and force it to be a short exact sequence

$$(16) \quad 0 \rightarrow H^1(E_K)(1) \rightarrow H^1(U_K)(1) \rightarrow \ker(j^*) \rightarrow 0.$$

General motivic folklore states that the splitting of (16) in \mathcal{MM}_K is equivalent to our above condition. Let us assume that this condition holds. The sequence (16) is dual to

$$(17) \quad 0 \rightarrow \text{im}(\delta) \rightarrow H^1(E_K, R_K) \rightarrow H^1(E_K) \rightarrow 0$$

where $\delta : H^0(R_K) \rightarrow H^1(E_K, R_K)$. Again according to general motivic folklore $\text{Ext}^1(\mathbb{Q}(0), \cdot)$ groups in \mathcal{MM}_K resp. $\text{Ext}^1(\mathbb{R}(0), \cdot)$ groups in a certain category of mixed Hodge structures \mathcal{MH}_R are naturally isomorphic to motivic cohomology resp. Deligne-Beilinson cohomology. Therefore applying those functors to (17) we get a compatible splitting

$$\begin{array}{ccccc} H_{\mathcal{M}}^2(E_K, R_K; \mathbb{Q}(2)) & \xleftarrow{s} & H_{\mathcal{M}}^2(E_K, \mathbb{Q}(2)) & \rightarrow & 0 \\ \downarrow r_{\mathcal{D}} & & \downarrow r_{\mathcal{D}} & & \\ H_{\mathcal{D}}^2(E_K, R_K; \mathbb{R}(2)) & \xleftarrow{s} & H_{\mathcal{D}}^2(E_K, \mathbb{R}(2)) & \rightarrow & 0. \end{array}$$

Using Galois descent we finally produce a splitting like (13).

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